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Project Report

Multi-Criteria Decision Analysis Using Interval Grey Numbers
Discipline: Methods of Optimization
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Summary

It is common in planning to require the balancing of multiple factors, or criteria. This is particularly true when the best option of each criteria of a particular problem is not possible and a trade-off must be made. To solve this problem, various techniques have been developed, under the group name "Multiple Criteria Decision Analysis". These techniques typically apply a weighting factor chosen by the planner, and various mathematical techniques to suggest one or more best alternatives. Additionally, it is common to require ranges of input criteria instead of specific or "crisp" numbers. Hence, another separate process, "interval grey numbers", which provides a mechanism for handling incomplete or missing information, is also included.

A brief description of multiple criteria decision analysis is provided and a non-exhaustive list of some current methods is shown. Additionally, three common methods from this list (in bold) are further explained in brief detail. Afterward, an explanation of interval grey numbers is also shown.

Finally, one of the methods, TOPSIS, is demonstrated and extended to use the methodology of interval grey numbers. The well-known example described in Zavadskas et al. (2008) is used, dealing with the selection of the best walls for housing. It is shown that the TOPSIS metho successfully reproduces the same results, and further expands the results using interval grey numbers to show pessimistic and optimistic views.

Overview – Multiple Criteria Decision Analysis

Description

Multiple Criteria Decision Analysis (MCDA) or Multiple Criteria Decision Making (MCDM) is the evaluation of an object or process on conflicting criteria. Typical examples of such criteria are cost, price, quality, safety, and efficiency. There is often the case that the idea solution is not available for each criterion. The solution process to this non-existing ideal solution problem is typically solved by the end user, who considers some criteria more important than others. Using this significance information, various approaches can be used to determine the most-suitable solution.

Typically Solved Problems

Below are example categories of typical problems solved by MCDM methodology.

- Choice
- Ranking
- Prioritization
- Resource Allocation
- Benchmarking
- Quality Management
- Conflict Resolution

List of MCDM Methods

Below is a non-exhaustive list of some documented MCDM methods, however many of them have not been formally extended for use with interval grey numbers. The bold items are discussed further.

- Aggregated Indices Randomization Method (AIRM)
- **Analytic hierarchy process (AHP)**
- Analytic network process (ANP)
- Best worst method (BWM)
- Characteristic Objects METhod (COMET)
- Choosing By Advantages (CBA)
- Data envelopment analysis
- Decision EXpert (DEX)
- Disaggregation – Aggregation Approaches (UTA*, UTAII, UTADIS)
- Dominance-based rough set approach (DRSA)
- ELECTRE (Outranking)
- Evaluation Based on Distance from Average Solution (EDAS)
- Evidential reasoning approach (ER)
- Goal programming (GP)
- Grey relational analysis (GRA)
- Inner product of vectors (IPV)
- Measuring Attractiveness by a categorical Based Evaluation Technique (MACBETH)
- Multi-Attribute Global Inference of Quality (MAGIQ)
- Multi-attribute utility theory (MAUT)
- Multi-attribute value theory (MAVT)
- New Approach to Appraisal (NATA)
- Nonstructural Fuzzy Decision Support System (NSFDSS)
- Potentially All Pairwise RanKings of all possible Alternatives (PAPRIKA)
- PROMETHEE (Outranking)
- **Simple Additive Weighting (SAW)**
- Stochastic Multicriteria Acceptability Analysis (SMAA)
- Superiority and inferiority ranking method (SIR method)
- **Technique for the Order of Prioritisation by Similarity to Ideal Solution (TOPSIS)**
- Value analysis (VA)
- Value engineering (VE)
- VIKOR method
- Fuzzy VIKOR method
- Weighted product model (WPM)
- Weighted sum model (WSM)
- Rembrandt method

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Overview – Interval Grey Numbers

Description

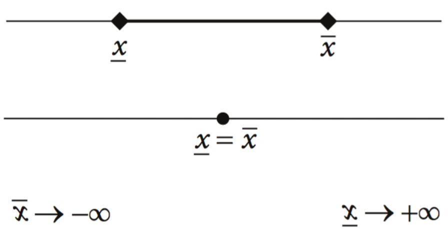
Interval grey numbers are used to represent partially known information. It is quite typical that a parameter's value is describe as a range rather than a single "crisp" number. The grey number system categorizes all values into three categories, by color:

- White – known information (exact number)
- Black – unknown information (no number)
- Grey – uncertain information (typically a range of numbers)

The following definitions and nomenclature are borrowed from an article by Dragisa (2012) which refers to Deng (1985, 1989, 1992), Liu et al. (1999) and Liu and Lin (1998, 2006).

Grey numbers are designated by $\oplus\, x$ and represent the range in which the value lies. This range, like mathematics, can be any form. Example: only upper limits, only lower limits, or both upper and lower limits. An upper limit is designated like $\overline{x}$ and a lower limit is designated as $\underline{x}$.

As earlier explained, the degree of greyness can be categorized into three categories. Below is a graphical explanation of each greyness level.



Interval Grey Number	Both upper and lower bounds are defined.
White Number	Upper and lower bounds are equal.
Black Number	Both upper and lower bounds tend toward infinity.

Mathematical Operations

The common mathematical operations per Deng 1992 and Liu, Lin 2006 are shown below.

Addition: $$\otimes\, x_1 + \otimes\, x_2 = [\underline{x}_1 + \underline{x}_2,\ \ \overline{x}_1 + \overline{x}_2]$$

Subtraction: $$\otimes\, x_1 - \otimes\, x_2 = [\underline{x}_1 - \underline{x}_2,\ \ \overline{x}_1 - \overline{x}_2]$$

Multiplication: $$\otimes\, x_1 \times \otimes\, x_2 = [\underline{x}_1 \underline{x}_2,\ \ \overline{x}_1 \overline{x}_2]$$

Division: $$\otimes\, x_1 \div \otimes\, x_2 = [\underline{x}_1, \overline{x}_1] \times [\frac{1}{\underline{x}_2}, \frac{1}{\overline{x}_2}]$$

Method Descriptions

Simple Additive Weighting (SAW)

The SAW method is probably the best-known method because it is the simplest. However, it has the limitation such that all criterion must have the same unit.

The scoring system uses the following equation where m is the number of alternatives, n is the number of decision criteria, w is the relative weight of a criterion, c is a criterion, and a is the performance value. The alternative is designated by A.

$$A_i^{\text{WSM-score}} = \sum_{j=1}^{n} w_j a_{ij}, \text{ for } i = 1, 2, 3, \ldots, m.$$

Analytic hierarchy process (AHP)

AHP was developed by Thomas L. Saaty and is based on psychology principles. It handles the problem-solving process by providing a comprehensive and rational framework for structuring the quantization and representation of the problem elements in relation to the overall goal and the possible solutions. The most common application fields are related to group decision making, in areas such as government, business, healthcare, and education.

The process can be split into the following steps:

1. **Hierarchy** – The decision problem is decomposed into a hierarchy of easier understood sub-problems, each of which is independent.
2. **Element Priority** – Each element is compared to each other element, two at a time, with respect to their impact on the element above them.
3. **Overall Priorities** – Combine the element priorities.
4. **Check** – Human verification of the overall priorities.
5. **Final Decision** – Generate a final decision based on the priorities and process.

Technique for the Order of Prioritization by Similarity to Ideal Solution (TOPSIS)

TOPSIS was originally developed in 1982 by Hwang and Yoon. It was later expanded again in 1987 and 1993. The basic principle is to find the shortest distance from the positive ideal solution and the longest distance from the negative ideal solution. The advantage of TOPSIS over some other methods is that it allows trade-off between criteria, allowing poor results from one criterion to be offset by the good results of another criterion.

The process can be split into the following steps:

1. **Evaluation Matrix** - Create a matrix with all criteria (n) and alternatives (m). The intersections of these criteria and alternatives will be X_{ij}.
2. **Normalize** - the matrix is normalized to prevent any criteria from having more natural weight than others, normally due to units.
3. **Apply Weight** – Apply the specified weight to each criterion, calculating the weighted sum.

4. **Worst/Best** – Determine, per criterion, the worst alternative (A_w) and best alternative(A_b).
5. **Ideal Comparison** – Determine the distance of the worst and best alternatives to the target alternative, in that criterion.
6. **Calculate Similarity** – Calculate how close the criterion is to the worst-case condition.
7. **Rank** – Using the combine similarity calculations, determine the final ranking. The highest score is the best alternative.

Results - Implementation of TOPSIS

The regular TOPSIS method described earlier will be demonstrated using the same well-known example described in Zavadskas et al. (2008). However, it will be modified to also include interval grey numbers. This example involves the selection of the best walls for housing.

Step 1 – Create the matrix of criteria

A matrix of (m) possible alternatives and (n) criteria is created. Each parameter is stored as a variable X_{ij} in the matrix of size m x n. The lower boundary of the criterion is represented by the interval grey number $\underline{\mathrm{X}}$ and the upper boundary is represented by $\overline{\mathrm{X}}$.

	Objectives									
	C1		C2		C3		C4		C5	
	Wall Durability		Thermal Resistance		Estimated Cost of m^2 walls		Weight of m^2 walls		Human work cost	
	cycles		W/m K		LTL		kg		hour/m^2	
Weight	0.21		0.33		0.26		0.26		0.11	
Opt.	max		min		min		min		min	
Alt.	$\underline{X1}$	$\overline{X1}$	$\underline{X2}$	$\overline{X2}$	$\underline{X3}$	$\overline{X3}$	$\underline{X4}$	$\overline{X4}$	$\underline{X5}$	$\overline{X5}$
A1	75	100	0.22	0.25	72.08	94.71	590	652	4.60	4.60
A2	75	100	0.22	0.25	89.01	100.93	596	625	4.60	4.60
A3	75	100	0.21	0.25	80.32	96.42	581	604	4.60	4.60
A4	25	25	0.24	0.27	67.76	98.10	455	479	4.55	5.01

Step 2 – Normalize the matrix

All criteria are normalized according the below equation, removing the effect of the units. Additionally, the values with optimization of “min” are multiplied by -1 to create a negative value so they become optimization problems of “max”.

$$r_{ij} = \frac{x_{ij}}{\sqrt{\sum_{i=1}^{m} x_{ij}^2}}, i = 1, 2, \ldots, m, j = 1, 2, \ldots, n$$

	Objectives									
	C1		C2		C3		C4		C5	
	Wall Durability		Thermal Resistance		Estimated Cost of m^2 walls		Weight of m^2 walls		Human work cost	
Weight	0.21		0.33		0.26		0.26		0.11	
Opt.	max		max		max		max		max	
Alt.	$\underline{r1}$	$\overline{r1}$	$\underline{r2}$	$\overline{r2}$	$\underline{r3}$	$\overline{r3}$	$\underline{r4}$	$\overline{r4}$	$\underline{r5}$	$\overline{r5}$
A1	1.558E-03	2.078E-03	-4.794E-01	-5.448E-01	-1.158E-03	-1.522E-03	-2.220E-04	-2.453E-04	-2.663E-02	-2.663E-02
A2	1.558E-03	2.078E-03	-4.794E-01	-5.448E-01	-1.430E-03	-1.622E-03	-2.242E-04	-2.351E-04	-2.663E-02	-2.663E-02
A3	1.558E-03	2.078E-03	-4.576E-01	-5.448E-01	-1.291E-03	-1.549E-03	-2.186E-04	-2.272E-04	-2.663E-02	-2.663E-02
A4	5.195E-04	5.195E-04	-5.230E-01	-5.884E-01	-1.089E-03	-1.576E-03	-1.712E-04	-1.802E-04	-2.634E-02	-2.900E-02

Step 3 – Apply Weight
All normalized criteria are multiplied by the normalized weight (ω_j), according to the below equations. This produces the weighted criteria (t_{ij}).

$$t_{ij} = r_{ij} \cdot w_j$$

$$w_j = W_j / \sum_{j=1}^{n} W_j$$

	Objectives									
	C1		C2		C3		C4		C5	
	Wall Durability		Thermal Resistance		Estimated Cost of m^2 walls		Weight of m^2 walls		Human work cost	
Alt.	$\underline{t1}$	$\overline{t1}$	$\underline{t2}$	$\overline{t2}$	$\underline{t3}$	$\overline{t3}$	$\underline{t4}$	$\overline{t4}$	$\underline{t5}$	$\overline{t5}$
A1	2.797E-04	3.730E-04	-1.352E-01	-1.537E-01	-2.574E-04	-3.382E-04	-4.933E-05	-5.451E-05	-2.503E-03	-2.503E-03
A2	2.797E-04	3.730E-04	-1.352E-01	-1.537E-01	-3.178E-04	-3.604E-04	-4.983E-05	-5.226E-05	-2.503E-03	-2.503E-03
A3	2.797E-04	3.730E-04	-1.291E-01	-1.537E-01	-2.868E-04	-3.443E-04	-4.858E-05	-5.050E-05	-2.503E-03	-2.503E-03
A4	9.324E-05	9.324E-05	-1.475E-01	-1.659E-01	-2.419E-04	-3.503E-04	-3.804E-05	-4.005E-05	-2.476E-03	-2.726E-03

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Step 4 – Determine Worst and Best
The greatest value (best) and lowest values (worst) for each criterion are located.

	C1	C2	C3	C4	C5
	Wall Durability	Thermal Resistance	Estimated Cost of m^2 walls	Weight of m^2 walls	Human work cost
	$A1$	$A2$	$A3$	$A4$	$A5$
Best (t_{bj})	3.730E-04	-1.291E-01	-2.419E-04	-3.804E-05	-2.476E-03
Worst (t_{wj})	9.324E-05	-1.659E-01	-3.604E-04	-5.451E-05	-2.726E-03

Step 5 – Determine Distance from best and worst
Each criterion is compared against the best and worst value from step 4. The intention is to identify those criteria closest to the best value and those criteria furthest from the worst criteria.

$$d_{ib} = \sqrt{\sum_{j=1}^{n}(t_{ij} - t_{bj})^2}$$

	Objectives									
	C1		C2		C3		C4		C5	
	Wall Durability		Thermal Resistance		Estimated Cost of m^2 walls		Weight of m^2 walls		Human work cost	
Best	$\underline{d1}$	$\overline{d1}$	$\underline{d2}$	$\overline{d2}$	$\underline{d3}$	$\overline{d3}$	$\underline{d4}$	$\overline{d4}$	$\underline{d5}$	$\overline{d5}$
A1	9.324E-05	0.000E+00	6.146E-03	2.458E-02	1.542E-05	9.623E-05	1.129E-05	1.647E-05	2.721E-05	2.721E-05
A2	9.324E-05	0.000E+00	6.146E-03	2.458E-02	7.587E-05	1.184E-04	1.179E-05	1.421E-05	2.721E-05	2.721E-05
A3	9.324E-05	0.000E+00	0.000E+00	2.458E-02	4.485E-05	1.023E-04	1.053E-05	1.246E-05	2.721E-05	2.721E-05
A4	2.797E-04	2.797E-04	1.844E-02	3.688E-02	0.000E+00	1.083E-04	0.000E+00	2.007E-06	0.000E+00	2.503E-04

$$d_{iw} = \sqrt{\sum_{j=1}^{n}(t_{ij} - t_{wj})^2}$$

	Objectives									
	C1		C2		C3		C4		C5	
	Wall Durability		Thermal Resistance		Estimated Cost of m^2 walls		Weight of m^2 walls		Human work cost	
Worst	$\underline{d1}$	$\overline{d1}$	$\underline{d2}$	$\overline{d2}$	$\underline{d3}$	$\overline{d3}$	$\underline{d4}$	$\overline{d4}$	$\underline{d5}$	$\overline{d5}$
A1	1.865E-04	2.797E-04	3.073E-02	1.229E-02	1.030E-04	2.221E-05	5.184E-06	0.000E+00	2.231E-04	2.231E-04
A2	1.865E-04	2.797E-04	3.073E-02	1.229E-02	4.256E-05	0.000E+00	4.682E-06	2.257E-06	2.231E-04	2.231E-04
A3	1.865E-04	2.797E-04	3.688E-02	1.229E-02	7.359E-05	1.610E-05	5.936E-06	4.013E-06	2.231E-04	2.231E-04
A4	0.000E+00	0.000E+00	1.844E-02	0.000E+00	1.184E-04	1.010E-05	1.647E-05	1.446E-05	2.503E-04	0.000E+00

Step 6 – Determine Similarity

A value of the similarity (s_{iw}) to the worst ideal is generated by comparing the distances of the worst (d_{iw}) and best (d_{ib}). High scores are preferred. The equation can be seen below.

$s_{iw} = d_{iw}/(d_{iw} + d_{ib})$

	Objectives									
	C1		C2		C3		C4		C5	
	Wall Durability		Thermal Resistance		Estimated Cost of m^2 walls		Weight of m^2 walls		Human work cost	
	$\underline{s1}$	$\overline{s1}$	$\underline{s2}$	$\overline{s2}$	$\underline{s3}$	$\overline{s3}$	$\underline{s4}$	$\overline{s4}$	$\underline{s5}$	$\overline{s5}$
A1	0.667	1.000	0.833	0.333	0.870	0.188	0.315	0.000	0.891	0.891
A2	0.667	1.000	0.833	0.333	0.359	0.000	0.284	0.137	0.891	0.891
A3	0.667	1.000	1.000	0.333	0.621	0.136	0.360	0.244	0.891	0.891
A4	0.000	0.000	0.500	0.000	1.000	0.085	1.000	0.878	1.000	0.000

Ranking

Finally, the sum of these similarity values is produced. Using these similarity values the best (highest score) and worst (lowest score) alternatives are identified. This process produces, using the interval grey numbers, both the lower boundary ($\underline{s}$) as well as the upper boundary ($\overline{s}$), providing additional insight about how the range of parameters affects the results. It can be seen that the ideal solution is either alternative A1 or alternative A3.

Alternatives	$\sum \underline{s}$	Rank	$\sum \overline{s}$	Rank
A1	3.576	1	2.412	2
A2	3.035	4	2.362	3
A3	3.540	2	2.604	1
A4	3.500	3	0.963	4

Conclusion

There are many methods for finding the best solution to a multi-criteria problem. This is particularly true when the best solution for each criterion of the problem is unavailable and a trade-off must be made. In these situations, methods such as SAW, AHP, and TOPSIS are able to quantify and standardize the process. Additionally, adding interval grey numbers to these processes allows for an understanding of how sensitive each criterion can be, providing outlook for possible pessimistic or optimistic planning

Testing of this methodology was performed using the TOPSIS method and the well-known example described in Zavadskas et al. (2008), involving the selection of the best walls for housing. As expected, the TOPSIS method is able to successfully reproduce the same results, suggesting that alternatives A1 and A3 are the best options, depending on pessimistic or optimistic planning.

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